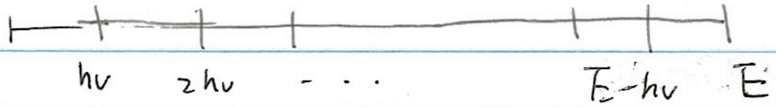
Pathria 1.8

$$\mathcal{E}(n) = nh\nu \; ; \quad n = 0, 1, 2, \ldots$$

$$\sum_{i=0}^{N} n_i = E/h\nu$$



We wish to place $N-1$ separators on segment of length $E/h\nu$. This will separate the segment into N small segments. This differs from $\binom{E/h\nu+1}{N-1}$ in that the separators here are allowed to over-lap.

There will be $N-1$ separators, together with $E/h\nu$ units of energy of size $h\nu$, the sum of separators and units of energy will be conserved and is equal to $(E/h\nu + N - 1)$, indepent of how we place the separators. Now, we consider the separators and units of energy on the same footing, we can do this because once we pick $N-1$ separators, and demand that the ones we don't pick are the units of energy, we get an accessible state of the system.

$$\underbrace{0\,|\,0\,|\,\cdots\,|\,0\,0\,0\,|\,0}_{\text{total of } N+E/h\nu-1 \text{ separators and } h\nu \text{ quanta}}$$

$$\Rightarrow \Omega(E, N) = \binom{E/h\nu + N - 1}{N-1}$$

$$\Omega(E,N) = \frac{(E/h\nu + N-1)!}{(N-1)!\,(E/h\nu)!}$$

$$S/k_B = \ln\Omega = \ln[E/h\nu + N-1]! - \ln[N-1]! - \ln[E/h\nu]!$$

$\Downarrow$ stirling approx. $\ln n! \simeq n\ln n - n$

$$\begin{aligned}\approx\ & (E/h\nu + N-1)\ln(E/h\nu + N-1) - \cancel{(E/h\nu + N-1)} \\ & - (N-1)\ln(N-1) + \cancel{(N-1)} \\ & - (E/h\nu)\ln(E/h\nu) + \cancel{(E/h\nu)}\end{aligned}$$

$$\begin{aligned}\frac{d\ln\Omega}{dE} = k_B^{-1}\frac{dS}{dE} &= \frac{1}{h\nu}\ln\left(E/h\nu + N-1\right) + \frac{\cancel{(E/h\nu + N-1)}}{\cancel{(E/h\nu + N-1)}}\cancel{\left(\frac{1}{h\nu}\right)} \\ &\quad - \frac{1}{h\nu}\ln(E/h\nu) - \cancel{\frac{1}{h\nu}} \\ &= \frac{1}{h\nu}\left[\ln(E/h\nu + N-1) - \ln(E/h\nu)\right] \\ &= \frac{1}{h\nu}\ln\left[\frac{E/h\nu + N-1}{E/h\nu}\right] = \frac{1}{k_B T}\end{aligned}$$

$$T = \frac{h\nu}{k_B}\,\frac{1}{\ln\left[\frac{E/h\nu + N-1}{E/h\nu}\right]}$$

$$\frac{E/h\nu + N - 1}{E/h\nu} = 1 + \frac{Nh\nu}{E} - \frac{h\nu}{E}$$

$$= 1 + \left(\frac{N}{E}\right) h\nu - \frac{1}{N}\left(\frac{N}{E}\right) h\nu$$

$$\approx 1 + \left(\frac{N}{E}\right) h\nu.$$

$$\Rightarrow \boxed{T \approx \frac{h\nu}{k_B} \frac{1}{\ln\left[1 + \left(\frac{N}{E}\right) h\nu\right]}}$$

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